LIMITING VISUAL MAGNITUDE AND NIGHT SKY BRIGHTNESS

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ABSTRACT. We review the theory of visual thresholds and applications to the limiting magnitude of a telescope and of the eyes, based on Schaefer's model with minor improvements. We apply our formulation to the Yerkes Observatory refractor and to naked eye observations at Mount Wilson Observatory. We reanalyze Bowen's telescopic observations at Mount Wilson by his approximate method and by our more elaborate theory. An extension of his method leads to a determination of the night sky brightness if the visual acuity of the observer is assumed to be average. Our more elaborate method allows a determination of the sky brightness, the visual acuity of the observer, and the average seeing during the observations.

1. Introduction

It has been of much interest in the past to ask what is the faintest stellar magnitude which can be seen by the naked eye with or without a telescope. Various formulae have been quoted for this purpose, but usually they have not taken into account the brightness of the night sky. For example, the traditional formula for the limiting visual magnitude m of a telescope can be put into the form

$$m = N + 5 \log D, \tag{1}$$

where D is the aperture of the telescope in centimeters and N is supposedly a constant. For many years, apparently starting with Young (1888, p. 470), the value $N = 7$ was used, and this was copied in many later books. Steavenson (1915a) drew attention to the disagreement between the traditional limiting magnitude formula (1) and observed values. Subsequent work showed that many observers can see stars fainter than equation (1) predicts. Various values of N have been proposed, ranging from 6.8 (Dimitroff and Baker, 1945, p. 44) to 8.7 based on an observation by Steavenson (1915b) of an 11.9 mag star using a 4.3 cm diameter objective. A still higher value of N would be needed to cover exceptional cases such as O'Meara's eyesight (discussed by Schaefer 1990). In the present author's opinion the best value for general use is perhaps that of Sinnott (1973) who proposed that $N = 7.7$. Improved methods of predicting m are desirable. One improvement was made by Bowen (1947), as will be discussed below, but not much use seems to have been made of his work. Major improvements were made by Schaefer and our own calculations described below follow his work with some relatively minor improvements. We have attempted to standardize our units by expressing telescope apertures, exit pupil diameters and eye pupil diameters in centimeters, illuminances in lux, and laboratory and sky background brightnesses in nanolamberts (abbreviated nL). This has the effect of making certain constants (such as N) have unfamiliar numerical values.

2. General formulae

We denote by i the illumination received directly from a star which is at the threshold of visibility to an observer, and by b the brightness of the night sky background. These are related by a formula $i = f(b)$. One important limitation in Bowen's work was his use of

$$i = f(b) = kb^{1/2}. \tag{2}$$

This may not be too bad an average if one must cover a range of 10^9 in background brightness with a single simple formula, but it is a poor approximation for small b, which is of interest in light pollution studies, because as b tends to zero i should tend to a constant threshold value. The formula does lead to simple results which we shall discuss in Section 6. Knoll, Tousey and Hulburt (1946) made a series of experimental determinations of the threshold for a point source seen against an illuminated background. They showed that their results could be represented by a relation of the form $f(b) = P(1 + kb)^{1/2}$, where $P = 1.076 \times 10^{-9}$ and $k = 1$ (by chance) if b is in nL and i is in lux. This equation does have the correct threshold behavior, but it only represents their data to within a factor 3 over their range of 10^9 in b. We shall not use this relation because better formulae can be obtained.

The problem of getting an appropriate relation between i and b was studied by Hecht (1934). On the basis of his chemical theory of vision he proposed a relation of the form

$$i = f(b) = C(1 + Kb^{1/2})^2, \tag{3}$$

where C and K are constants. The chemical theory of vision used by Hecht to justify using this form of equation is no longer accepted physiological theory, but the form of equation seems to describe the observed relationships remarkably well. On the basis of the observations by Knoll, Tousey and Hulburt (1946) of threshold intensities Hecht (1947) proposed using two relations of the same form, one for faint illuminances when the rods of the retina are dominant, and one for bright illuminances when the cones of the retina dominate. The numerical equations given by Hecht were used by Weaver (1947), Garstang (1986) and Schaefer in studies of the naked eye visibility of stars under various conditions of sky brightness.

Blackwell (1946) described a very large set of laboratory naked eye binocular observations of threshold contrast as a function of b (about 2 million observations were made and 450,000 analyzed). We use the final results in his Table VIII, which were based on about 90,000 observations by seven observers whose average age was about 23 years. Seven circular disks of various diameters were used as stimuli against a large background whose brightness could be varied from 10^9 nL down to 10 nL. The experiments determined the threshold contrast for seeing a disk against the background. Tousey and

Hulburt (1948) modified Blackwell's data by changing from threshold contrasts to absolute thresholds, changing the units, and doubling the values of i to change the threshold criterion from a 50% probability of detection to a 98% probability of detection. We further changed the units of i to lux and the units of b to nanolamberts. All the photometry was expressed in terms of the photopic response curve. We applied a correction to the observations of Blackwell in the scotopic region to allow for the difference of color temperatures between his sources and those of Knoll, Tousey and Hulburt. (The latter authors had 2360^o K; Tousey and Hulburt stated that Blackwell had 2850^o K.) Some additional experiments were performed with zero background brightness (Blackwell 1946, Table IV). We converted these measurements to values of i in lux. Finally we applied small systematic corrections to Blackwell's data for each background brightness separately so that for effectively point sources Blackwell's data would agree with Knoll, Tousey and Hulburt. The effect of this is to ensure that the Knoll, Tousey and Hulburt data were used for point sources and the contrast ratios measured by Blackwell were used for larger sources.

We wanted to use equations of the basic form (3), but we sought to generalize the equations to include the effect of stimulus size θ as described by Blackwell's data. In astronomical applications θ is the seeing disk diameter. Blackwell gave θ in arc minutes. We also wanted to smooth out the transition between scotopic and photopic formulae to eliminate the discontinuity in the use of two separate Hecht type formulae without losing much of the character of the two formulae. After some trials we adopted the following formulae:

$$i_1 = c_1(1 + k_1 b^{1/2})^2(1 + \alpha_1\theta^2 + y_1 b^{z_1}\theta^2) \tag{4a}$$

$$i_2 = c_2(1 + k_2 b^{1/2})^2(1 + \alpha_2\theta^2 + y_2 b^{z_2}\theta^2) \tag{4b}$$

$$i = i_1 i_2/(i_1 + i_2) \tag{4c}$$

These three equations represent the function $f(b)$. Unlike earlier authors we must evaluate both i_1 and i_2 for all values of b. Equation (4c) is a purely mathematical artifact which we have introduced to provide a smooth transition from the response function of the rods to that of the cones. This smooth transition seems apparent in the final results of Knoll, Tousey and Hulburt. For small b we find that i_1 is appreciably smaller than i_2 so that $i = f(b)$ does not differ greatly from the scotopic threshold i_1, while for large b we find that $i_1 \gg i_2$ so that $i = f(b)$ is nearly equal to the photopic threshold i_2. We tried some other formulae, but we did not find one better than (4c). Then we took our combination of Knoll, Tousey and Hulburt's data and Blackwell's data, and determined the best fit of our formulae. We found that the best fit was given by omitting the data for $b = 10^9$ nL and $\theta = 360$ arc minutes. We obtained $c_1 = 3.451 \times 10^{-9}$, $c_2 = 4.276 \times 10^{-8}$, $k_1 = 0.109$, $k_2 = 1.51 \times 10^{-3}$, $y_1 = 2.0 \times 10^{-5}$, $\mathrm{y}_2 = 1.29 \times 10^{-3}$, $z_1 = 0.174$, $z_2 = 0.0587$, $\alpha_1 = 2.35 \times 10^{-4}$, $\alpha_2 = 5.81 \times 10^{-3}$. The values $b = 10^9$ nL and $\theta = 360'$ are rather extreme, and not of interest for our planned calculations, but in spite of our omitting them from the fitting our formulae do give reasonably good predictions of i if either $b = 10^9$ nL or $\theta = 360'$ or both. To convert i into magnitudes we use the relation (Allen 1973 p. 197)

$$m = -13.98 - 2.5 \log i \tag{5}$$

where i is expressed in lux.

Strictly speaking, our formulae apply to observers aged about 23 years. To apply our formulae to observers of other ages we need a formula for the diameter of the eye pupil p as a function of the age of the observer A and the brightness of the sky background b. We considered the papers by Kadlecova et. al. (1958) and Kumnick (1954). From these we estimated the variation of pupil diameter with age for dark backgrounds assuming a linear relation. From Kumnick's data we estimated (including the filter factor) that her data for bright backgrounds was obtained with a background of $b = 2.5 \times 10^6$ nL, and we obtained the variation with age as a linear relation. Finally we used the form of relationship used by Moon and Spencer (1944) to combine the dark and bright relationships into a single formula. We obtained

$$p = 0.534 - 0.00211A - (0.236 - 0.00127A) \tanh (0.40 \log b - 2.20) \qquad (6)$$

Our result agrees quite closely with diameters obtained by Schaefer (1990, equation (5)) for ages between 20 and 90. Although not needed for most light pollution work, our formula will work for skies as bright as daylight. Individual observers may show significant deviations from the average represented by equation (6). After the present paper had been completed we found the results of I. E. Loewenfeld (quoted by MacRobert 1992). If we had included her results in our averages we would have obtained an average dark sky pupil diameter 0.2 or 0.3 mm smaller at all ages from 20 to 90. Our results would be only slightly affected.

Schaefer (1990) gave an extensive discussion of additional factors which must be included to obtain accurate results. All the correction factors are defined in the sense that i_1 and i_2 must be multiplied by the appropriate factors to give the correct threshold. The correction factors must be determined separately for i_1 and i_2. The appropriate corrections must also be applied to b. The factors are: (a) a factor F_b to take into account that one eye is used in telescopic observations, while binocular vision was used in obtaining the relations between i and b, (b) a factor F_e to allow for extinction in the terrestrial atmosphere, (c) a factor F_t to allow for the loss of light in the telescope, F_t being the reciprocal of the transmission t through the telescope and eyepieces, (d) a factor F_p to allow for the loss of light if the telescope exit pupil is larger than the eye pupil, (e) a factor F_a to take into account the ratio of the area of the telescope to that of the naked eye, (f) a factor F_m to allow for the reduction of the sky brightness by the telescope magnification, (f) a factor F_{SC} to take the Stiles-Crawford effect into account, (g) a factor F_c to allow for the difference in color between the laboratory sources used in determining the relationships between i and b and the stars being observed, and (h) a factor F_s to allow for the acuity of any particular observer, defined so that $F_s < 1$ leads to a lower threshold i and therefore implies an eye sensitivity higher than average.

For telescopic observations we must replace the image size θ by $M\theta$ in our equations (4), where M is the magnification of the telescope. Then Schaefer's F_r is not needed because we have already included the image size in our equations (4). Schaefer also gave an experience correction. This can be used if desired; we did not use it, the effect of experience is included in our F_s. In calculating the correction factors we have followed Schaefer closely, except that for F_{SC} we used the formulation of Moon and Spencer (1944), which gives results almost the same as Schaefer's formulation (after correction for an important misprint in Schaefer's work: his formulae for F_{SC} are the inverses of the correct formulae, so that his formulae give $1/F_{SC}$). It is perhaps a misnomer to call F_a and F_m correction factors because they are primary factors allowing a telescope to see objects fainter that the eye can see. In fact $F_a = p^2/D^2$, where D is the aperture of the telescope, and $F_m = M^2$. We refer the reader to Schaefer's paper for detailed discussion and formulae for all the other factors.

The factors can be combined as

$$F = F_b F_e F_t F_p F_a F_{SC} F_c F_s \quad (7)$$

$$G = 1/(F_b F_t F_p F_a F_m F_{SC} F_c) \quad (8)$$

The individual correction factors must be calculated separately for the scotopic and photopic cases. We use additional subscripts 1 for scotopic and subscripts 2 for photopic correction factors. Then c_1 must be replaced by $F_1 c_1$ in equation (4a) and c_2 must be replaced by $F_2 c_2$ in equation (4b). We must replace b by $G_1 b$ in equation (4a), b by $G_2 b$ in equation (4b), b by $G_1 b$ or $G_2 b$, as appropriate, in equation (6), and θ by $M\theta$ in equations (4a) and (4b). Note that F_m does not occur in F and that F_e and F_s do not occur in G.

3. Naked eye observations

Our method can be applied to observations made with the naked eye. Some of the corrections discussed above do not apply to the eyes, and a different formula is needed for F_a. If p_0 is the pupil diameter used by the average of the Knoll, Tousey, Hulburt and Blackwell observers, who are assumed to have been age 23, and p is the pupil diameter used by some other observer, calculated from equation (6) above, then $F_a = p_0^2/p^2$. The correction factors are given by

$$F = F_a F_{SC} F_c F_e F_s \quad (9)$$

$$G = 1/(F_a F_{SC} F_c) \quad (10)$$

They must be calculated for scotopic and photopic conditions separately, and applied in the manner described above, with $M = 1$.

4. Application to Yerkes Observatory

We felt the need for a check on our method. Barnard (1913) observed the long-period variable star AG Cygni, and his observations showed that the limiting magnitude of the Yerkes 102 cm refractor was about 17.0. These observations make a valuable benchmark even today, for not only were they made by an experienced observer with the largest refractor in the world, but they were made in the days when light pollution at Yerkes Observatory was negligible. He observed AG Cygni on 80 nights between November 1910 and April 1913. He missed a maximum which occurred in January 1912, and so the period is one half of what he thought. We used a period 292 days (slightly shorter than the modern value) which fits Barnard's observations and replotted his observations with this period. A very presentable light curve appeared, though with a large scatter of the observations, the light curve being very similar in form to that of R Aur (Isles and Saw 1987). The curve shows a nice minimum at about magnitude 17.0, in agreement with Steavenson (1915a). We estimate the uncertainty as two or three tenths of a magnitude. There are other difficulties in Barnard's work, including uncertainities of the comparison star magnitudes and the general effects of the rather poor atmospheric conditions at Yerkes Observatory.

We used our program described above to calculate the limiting magnitude of the Yerkes refractor. We assumed age 55 for Barnard, magnifications 460 and 700 which were used by Barnard, an extinction of 0.32 magnitude, a telescope transmission of 0.61 based on estimates of the reflection losses and absorption in the objective and eyepieces, and other parameters with their usual values. We verified by using our programs that light pollution would have been negligible. Barnard's observations were made at a time when the sunspot counts were very low, so we assumed a solar minimum night sky background brightness of 55 nL. We assumed limiting magnitudes of 17.0 with $M = 460$ and 17.1 with $M = 700$, based on Barnard's statements on several occasions that he could not see AG Cygni with $M = 460$ but he could glimpse it with $M = 700$. The best fits we obtained were with $F_s = 0.65$, $\theta = 1.5''$ and $F_s = 0.68$, $\theta = 1''$, there being little to choose between these. The seeing is not well determined, its value depends on the difference we assume between the limiting magnitudes for $M = 460$ and $M = 700$. The limiting magnitude for $M = 700$ is quite close to the optimum value of M, beyond which for larger M the limiting magnitude becomes brighter; for $\theta = 1.5''$ the optimum from our calculations is $M = 800$. The value of F_s significantly smaller than unity is a confirmation of the well known above average eyesight of Barnard. The fit we have obtained seems to confirm the broad correctness of our model.

5. Application to Mount Wilson

The simple formulae mentioned at the beginning of this paper made no reference to the increased brightness of the night sky background due to light pollution. This was of no importance in the calculations above on Yerkes Observatory. However, at Mount Wilson Observatory light pollution is substantial, and this affects the limiting magnitude which can be observed. We calculated the limiting magnitude for a naked eye observer aged 40 at Mount Wilson, using the night sky brightnesses which we have calculated (Garstang 1999). The results are given in Table I of that paper. The results illustrate very well the steady worsening of the night sky at Mount Wilson.

6. Bowen's approximate formula

Another simple application of the above method is to the derivation of a simple formula given by Bowen and obtained by him in a direct way. We put $F_{SC} = 1$, $F_c = 1$ and $F_b = 1.41$ for a single eye. We use the appropriate formulae for F_p so that we take account of whether the exit pupil is larger or smaller than the eye pupil. Finally, we use equation (2) instead of our equations (4). The value of k is that given by Langmuir and Westendorp (1931), which when changed into our units and doubled to give a detection probability of 98% is $k = 1.25 \times 10^{-9}$.

We write $E = D/M$ for the diameter of the exit pupil of the telescope. Putting in the formulae for the correction factors we find after some algebra that

$$E > p \quad m = C + 5 \log D - 5 \log E + 2.5 \log p + 1.25 \log t, \tag{11}$$

$$E \leq p \quad m = C + 5 \log D - 2.5 \log E + 1.25 \log t. \tag{12}$$

where C is now defined by

$$C = 8.09 - 2.5 \log p - 1.25 \log b - x - 2.5 \log F_s \tag{13}$$

and x is the extinction in magnitudes suffered by the starlight. Bowen did not include t in his formulae, but he did mention its importance for the 152 cm telescope at Mount Wilson. If we put $t = 1$ and $E = D/M$ in equation (12) we get

$$E \leq p \quad m = C + 2.5 \log D + 2.5 \log M \tag{14}$$

the formula given by Bowen. Bowen did not give the formula (13) for C, and hence he did not obtain any value of b.

It is important to note that C is not necessarily a constant. Even if we neglect variations of p, C depends on b and on x. C may be treated as a constant for a given observatory if an average sky brightness is assumed and an average extinction is valid, the averages being taken over the part of the sky of interest (usually not excessively far from the zenith). That is essentially what Bowen did. We have reanalyzed the data given by Bowen (1947, Fig. 1). He gave limiting magnitudes observed at the Mount Wilson Observatory using three telescopes and various magnifications. Two were refractors with coated objectives having $D = 0.84$ cm and $D = 15$ cm. The third was a reflector having $D = 152$ cm (the famous 60-inch reflector, which is a 3 mirror cassegrain). We estimated the transmissions of the telescopes, assuming non-reflection coatings on all air-glass lens interfaces including the eyepieces, and allowing for the losses at the three reflections and the loss by secondary support obstruction in the reflector. We obtained $t = 0.92$ for the refractors and $t = 0.58$ for the reflector. The adopted values of t are not critical. We then calculated the quantity $m - 5 \log D - 1.25 \log t$, and plotted it against $-\log E$. The resulting diagram (Fig. 1) is very similar to Bowen's diagram, but the points for the 152 cm telescope are closer to those for the other telescopes. It shows a nearly linear relationship. We omitted the two points with $-\log E = 0.99$ and 1.28 because we believe that the former is severely affected by seeing and the latter is severely affected by the failure of the Langmuir-Westendorp relation at small b. We used least squares to fit two straight lines of slopes 2.5 and 5 according to equations (11) and (12) to the remaining points and obtained the constant in equation (13) as $C = 5.64$. We used equation (6) to estimate p, taking $A = 48$ for Bowen in 1947 and examining the variation of p over various background brightnesses b. We adopted $p = 0.60$ cm. We took $x = 0.20$ as an average value of the extinction at scotopic wavelength 510 nm within about 40° of the zenith. We assumed that $F_s = 1$. Equation (13) then allowed us to calculate b,

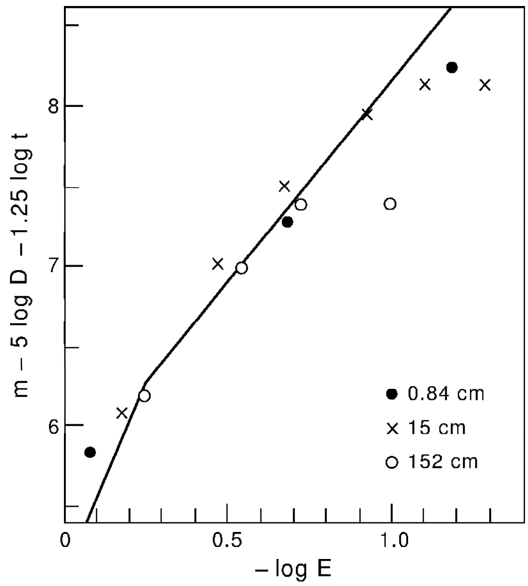

Fig. 1. Bowen's diagram of limiting visual magnitude as a function of the diameter of the exit pupil E, replotted with changed units and with the inclusion of the correction (t) for telescope losses. The lines are the best fit of equations (11) and (12), omitting the points with $-\log E = 0.99$ and 1.28. The slope changes at the value $E = p = 0.60$ cm.

with the result $b = 170$ nL. This value is of course subject to uncertainty from many causes, including the fitting process, the failure of Langmuir and Westendorp's relation for small b, whether their value of k (as modified by us) is appropriate for the threshold criterion used by Bowen, whether $F_s = 1$ is a correct description of Bowen's visual acuity, and whether our chosen value of p is a fair average for Bowen's eye under various magnifications and hence background brightnesses. We estimate the uncertainty to be at least a factor of 2. It should be noted that, according to equation (12), b and F_s cannot be determined independently by Bowen's method. We actually have determined $bF_s^2 = 170$ nL and assumed that $F_s = 1$.

7. More accurate analysis of Bowen's observations

It is interesting that we have been able to make an estimate of the night sky brightness at Mount Wilson in 1947: this is of interest for our work on light pollution because there are few published measures of the brightness in the literature. It is therefore worth while making a more accurate analysis of Bowen's observations. We took the theory described in Section 2 above, and calculated all the correction factors. We assumed an extinction of 0.17 in V magnitudes for an average zenith distance of perhaps 30^o. We assumed a star color of B − V = 0.7. Consideration shows that there are 3 significant unknowns

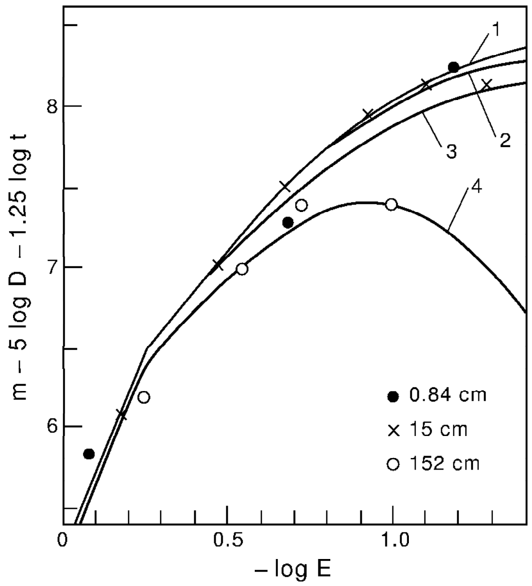

Fig. 2. Bowen's observed points are plotted for telescopes with $D = 0.84$ cm, $D = 15$ cm and $D = 152$ cm. Curve 1 is for $D = 0.84$ cm with any value of the seeing. The curve for $D = 15$ cm and $\theta = 0$ is indistinguishable from curve 1. Curve 2 is for $D = 15$ cm and $\theta = 1''.5$. Curve 3 is for $D = 152$ cm and $\theta = 0$. Curve 4 is for $D = 152$ cm and $\theta = 1''.5$. Curves 1, 2 and 4 were derived from parameters derived from a least-squares fitting to all the points in a single set of calculations. The parameters were $F_s = 0.69$, $\theta = 1''.5$ and $b = 330$ nL. Curve 4 shows that the seeing is primarily determined by the observation with the 152 cm telescope and the highest magnification used ($M = 1500$).

in our problem (i.e., quantities which we cannot guess), the brightness b, the seeing θ and the factor F_s. We performed calculations for ranges of values of b, θ and F_s and determined the values of b, θ and F_s which minimized the least squares deviations between our calculated values and Bowen's observations for all three telescopes in a single calculation. Our final results are $b = 330$ nL, $s = 1''.50$ and $F_s = 0.69$. Our theory not only produces a value of the night sky brightness, but it also gives a value of the seeing at Mount Wilson during Bowen's observations and an estimate of F_s. Because $F_s < 1$ it shows that Bowen had a fainter than average threshold. This may be due to above average retinal sensitivity, to his scientific experience, and possibly to an above average eye pupil size. Other factors such as errors in the comparison star magnitudes may also contribute. If we accept the value $b = 330$ nL as the best attainable, we may ask if the value is reasonable. We note that when Bowen was observing solar activity was rising towards maximum. We do not know the dates of his observations, but in late 1946 and early 1947 the sunspot number averaged roughly 120 (Waldmeier 1961). The correlation of Walker (1988) leads to an estimate of 75 nL for the natural night sky background brightness. There is a residual of 255 nL which we attribute to light pollution at Mount Wilson from the Los Angeles basin. This estimate is of importance as a check on our light pollution calculations for Mount Wilson, and it is discussed in Garstang (1999). We conclude that the study of limiting visual magnitudes can give useful information on night sky brightnesses in cases where there is severe light pollution.

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